Boolean Algebra

Def^n Let B be a non-empty set with two binary operations $+$ and $*$, a unary operation $'$ and two distinct elements 0 and 1. Then B is called a Boolean Algebra if the following axioms hold for any $a, b, c \in B$.

(1) Commutative laws –
'+' & '*' are comm. i.e.
$a+b = b+a$ & $a*b = b*a$ $\forall a, b \in B$

(2) Identity laws –
$\forall a \in B$, we have –
$a+0 = a$ & $a*1 = a$

(3) Distributive laws –
We have –
$a+(b*c) = (a+b)*(a+c)$; '+' dist^r over '*'
$a*(b+c) = (a*b)+(a*c)$; '*' dist^r over '+'
for any $a, b, c \in B$

(4) Complement laws –
For each $a \in B$, $\exists$ an element a' in B s.t.
$a+a' = 1$ & $a*a' = 0$.

we denote Boolean ~~algebra~~ algebra by $(B, +, *, ', 0, 1)$.

$0 \to$ zero element (identity for $+$)

$1 \to$ unit element (identity for $*$)

We may also use symbol '$\vee$' in place of '$+$' and '$\wedge$' in place of '$*$' i.e.

$a+b = a \vee b$; $a * b = a \wedge b$

$a' = \bar{a}$

<u>Ex</u> $S \to$ non-empty set

$P(S) \to$ set of all subset of S

i.e. Powerset of S.

Then $P(S)$ is Boolean Alg. w.r.t. union & intersection as '$+$' & '$*$'.

$' \to$ complement

$\phi \to 0$; $S \to 1$

<u>Sol^n.</u> (1) Comm. Laws —

We know

$A \cup B = B \cup A$ & $A \cap B = B \cap A$

$\forall A, B \in P(S)$

(2) Identity Laws —

$\phi, S \in P(S)$

$A \cup \phi = A$ & $A \cap S = A$ for any $A \in P(S)$

Then ϕ & S are identities of $P(S)$

3. <u>Distr Laws</u> we know that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

& $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

for any $A, B, C \in P(S)$

4. <u>Complement Laws</u> for any $A \in P(S)$,

we have $S - A \in P(S)$ s.t.

$A \cup (S - A) = S$ and $A \cap (S - A) = \phi$

$\therefore A' = S - A \in P(S)$

Hence $(P(S), \cup, \cap, ', \phi, S)$ is a Boolean Alg.

Ex. $B = \{0, 1\}; +, *, '$ s.t.

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	0	1

'	0	1
	1	0

B is Boolean Algebra

Ex. $B = \{1, 2, 5, 7, 10, 14, 35, 70\}$

Define →

$a + b = \text{lcm}(a, b)$

$a * b = \gcd(a, b)$ $\quad \forall a, b \in B$

$a' = \dfrac{70}{a}$

Then B is a Boolean Alg. with 1 as zero element & 70 as unit element